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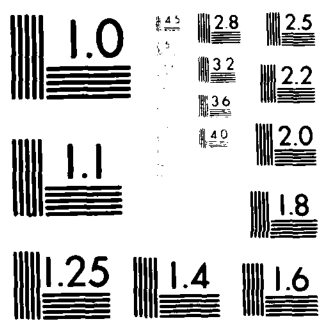

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Poisson Flows on Markov Step Processes

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December 1979

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POISSON FLOWS ON MARKOV STEP PROCESSES

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ABSTRACT

A Markov step process Z equipped with a possibly non-denumerable state space χ can model a variety of queueing, communication and computer networks. The analysis of such networks can be facilitated if certain traffic flows consist of mutually independent Poisson processes. Accordingly, we define the multivariate counting process $N=(N_1, N_2, \ldots N_c)$ induced by Z; a count in N_i occurs whenever Z jumps from $x \in X$ into a (possibly empty) target set $\Gamma_x^{(i)}$. We study N through the infinitesimal operator A of the augmented Markov process W=(Z,N), and the integral relation connecting A with the transition operator T_t of W. It is then shown that N_i is expressed in terms of a non-negative valued function r_i defined on χ; $r_i(x)$ may be interpreted as the expected rate of increase in N_i, given that Z is in sate x.

For univariate N (i.e., c=1), we show that N is Poisson iff (a) $E[r(Z(t))]$ is constant and (b*) $E[r(Z(t))|N(t)]=E[r(Z(t))]$ for each $t \geq 0$. These "local" conditions are weaker than the usual global sufficiency criteria, which moreover require stationarity (of Z) and independence (of N(t) and Z(t)). N sub i r sub i

A multivariate N is Poisson (i.e., composed of mutually independent Poisson streams N_i) if each $r_i(Z)$ is stationary, and $r_i(Z(t))$ is independent of N(t) for each i and each $t \geq 0$. The latter is already much less restrictive than the independent of N(t) and Z(t), but ~~we~~ ~~are~~ IT IS able to find even weaker hypotheses which are both necessary and sufficient for N to be Poisson.

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0. INTRODUCTION

In earlier papers, the authors [1][13][14] and others [10][15] [7][11] have studied traffic processes in Markov feedback queueing networks, and more generally, traffic processes appropriately defined on regular Markov processes with a denumerable state space. One application has been to Jackson networks (exponential servers, random instantaneous routings, and independent Poisson exogenous arrival streams) in equilibrium; we have shown that certain traffic processes consist of mutually independent Poisson streams, while others--those within loops or cycles--cannot be Poisson. The same type of result extends to more general queueing networks, in the sense that we have found some necessary and sufficient conditions that traffic processes are Poisson [13] [14]. However, we know of no prior result asserting complete necessity conditions for multivariate counting processes induced by a Markov step process.

We are now able to extend the theory to apply to a Markov step process Z on a non-denumerable state space χ, and to cover simultaneous sufficiency and necessity conditions for both univariate and multivariate counting processes N induced by Z. As in [13], the notion of traffic is abstracted to the less restrictive concept of jumps from originating to target sets in the state space; the number of such jumps constitutes the counting process under investigation. Moreover, we introduce a unified methodology that simplifies prior approaches.

Melamed [13] has summarized arguments in the queueing system literature purporting to show that certain traffic streams are Poisson [1][3][7][10][15]; in each instance, the proof is equivalent to demonstrating the independence of $N(t)$ and $Z(t)$ for each $t \geq 0$ under the

additional hypothesis that Z is in equilibrium. It is now known [13] that weaker conditions suffice for univariate or multivariate N. Further, a necessity result [13] exists for univariate N. In our work, we shall find conditions that are simultaneously necessary and sufficient for both univariate and multivariate N.

The "global" nature of the independence between N(t) and Z(t) fails to discriminate between parts of Z relevant to N, and those aspects of Z having no influence on N at all. It is thus intuitively plausible that the usual sufficiency conditions can be improved, and necessity also considered. It turns out that "local" criteria for a Poisson N can indeed be derived. To this end, we augment the Markov step process Z to incorporate the traffic N via the new Markov step process W=(Z,N). We study N through the infinitesimal operator A of W, using the integral relation connecting A with the transition operator T_t. In this fashion, we elucidate the properties of a non-negative function r (for univariate N) defined at least on the domain of A. We can interpret r(x) as the jump rate of N, given that Z is in state x. For univariate N, we find N to be Poisson iff (a) the expectation $E[r(Z(t))]$ is constant, and (b*) the conditional expectation $E[r(Z(t))|N(t)]=E[r(Z(t))]$ for each $t\geq 0$.

For multivariate $N=(N_1,N_2,\ldots,N_c)$, it is proved in Section III that extensions of (a) and (b*) suffice for the N_i to be mutually independent Poisson streams. However, an even less restrictive set of conditions are discovered to be both sufficient and necessary in the same regard. Another set of conditions is shown to be equivalent to the latter set. In each of these cases, the specified conditions reduce to (a) and (b*) when c=1, i.e., when N is actually univariate.

We stress that each of the results quoted above are applicable to Markov step processes on denumerable as well as non-denumerable state spaces. This generalization over [3][4][7][10][11][13][15] does not entail any sacrifice in the strnegth of the conclusions obtained, nor does it require a more complicated analytic methodology. Although it is not customary to contemplate pure jump Markov queueing or storage systems on non-denumerable state spaces, it is easy to visualize models of systems that are essentially non-denumerable. These might involve variable magnitude or partial service, as well as processing of units (e.g., messages or rainfall) of random size. As for multivariate N, we remark that its analysis has already led to the construction of a maximal decomposition for Jackson networks [1], and that there may well be new applications to respective counting processes generated by multiple classes of service and/or customers.

I. PRELIMINARIES

Our intent is to analyze a counting process N induced by a Markov jump process Z. We shall need some standard notions applicable to such processes; these are stated without detailed explanation, but with sufficient references to aid the reader unfamiliar with the material. For Z, we suppose a state space χ equipped with a σ-algebra $\mathcal{E}$ and the discrete topology. In the terminology of Dynkin ([6], p. 93), Z is required to be a step process. The weak infinitesimal operator A of Z ([6], Section I.6) is a (possibly unbounded) transformation with representation ([6], Section V.2)

$$(Ag)(x) = -q(x)g(x) + q(x)\int_{\chi} Q(x,dy)g(y), \quad x\epsilon\chi. \tag{1.1}$$

Here A is a mapping from a subset of $\mathcal{B}$ into $\mathcal{B}$, $\mathcal{B}$ being the set of bounded real-valued functions on $(\chi,\mathcal{F})$. Moreover, Q is measurable separately in each variable, with $Q(x,\chi)=1$ for each $x\epsilon\chi$. We remark that $Q(x,\Gamma)$ is regarded as the probability that a jump, starting from x, will take Z into Γ. The function q is the jump rate, q(x) being the parameter of the exponentially distributed sojourn time, given that Z is at x.

To preclude trivialities, we specify that there be no absorbing states; then $0<q(x)<\infty$ for the step process. Under the stronger hypothesis--met by most practical systems--that $\sup[q(x)]<\infty$, A becomes an endomorphism which is identical with the strong infinitesimal operator ([12], p. 643). Otherwise, we are forced to simply assume that the indicators $g\epsilon\mathcal{B}$ to appear hereafter lie in the domain of the weak infinitesimal operator A. This cannot be guaranteed, as is indicated by the example of the departure process from the simple $M/M/\infty$ queue.

Our principal concern is a counting process N generated by Z. A count occurs whenever Z jumps from x directly into Γ_x where Γ_x is a (possibly empty) measurable subset of χ such that $x\not\epsilon\Gamma_x$. In terms of the notation established for (1.1), $Q(x,\Gamma_x)$ is the probability that a next jump of Z increments N. To place this notion in a suitable context, augment Z to the new Markov process

$$W = (Z,N) \tag{1.2}$$

on a state space $\chi\times N^+$, where N^+ is the set of non-negative integers. On the new state space, the σ-algebra is that generated by $\Gamma\times\{k\}$, $\Gamma\epsilon\mathcal{F}$, $k\epsilon N^+$. The infinitesimal operator $\tilde{A}$ for W is defined by (1.1) with the same rate function q, but with Q replaced by

$$\tilde{Q}((x,n),\Gamma\times\{k\}) = \begin{cases} Q(x,\Gamma)\,\delta_{n+1,k} & \Gamma\subset\Gamma_x \\ Q(x,\Gamma)\,\delta_{n,k} & \Gamma\cap\Gamma_x=\phi \end{cases} \tag{1.3}$$

where δ is the Kronecker delta. If we take $\tilde{g}(x,n) = g(x)$ and use (1.3) in (1.1), we see that $(A^k g)(x) = \sum_n (\tilde{A}^k \tilde{g})(x,n)$ for each $k=1,2,\ldots$; hence, by the exponential formula of semi-group theory the transition probabilities P and $\tilde{P}$ are related by

$$P(t,x,\Gamma) = \sum_{n=0}^{\infty} \tilde{P}(t,(x,0),\ \Gamma\times\{n\}) = \tilde{P}(t,(x,0),\Gamma\times N^+). \tag{1.4}$$

The projection of W on χ then has the same probability structure as Z, provided that the initial probabilities are related by

$$\psi_0(\Lambda) = \varphi_0(\Lambda\times\{0\}), \tag{1.5}$$

where ψ_t and φ_t are the probability measures of $Z(t)$ and $W(t)$, respectively. In (1.5), $N(0)=0$ almost surely is assumed as a matter of convenience.

Thus, W is an augmentation of Z by the counting process N. Moreover, the argument of Theorem 1 in [1] is applicable to show that W retains the Markov property as well as the step process behavior. In fact, we have

Theorem 1.1: W is a Markov step process whose weak infinitesimal operator $\tilde{A}$ is represented by (1.1), with functions $\tilde{Q}$ given by (1.3) and $\tilde{q}(x,n)=q(x)$. If q is bounded, $\tilde{A}$ is an endomorphism on the bounded measurable functions on $\chi\times N^+$; if $g\varepsilon\mathcal{D}_A$ (the domain of A), and h is a bounded function on N^+, then $gh\varepsilon\mathcal{D}_{\tilde{A}}$. Under the hypothesis (1.5)

$$P[\bigcap_1^n\{X(t_k)\varepsilon\Gamma_k\}] = \tilde{P}[\bigcap_1^n\ \{W(t_k)\varepsilon(\Gamma_k\times N^+)\}] \tag{1.6}$$

for all collections of t_k and measurable Γ_k. □

We now complete this Section by listing several standard relations ([6], Chapter 2) appropriate to our purpose. Starting from the transition probability, we can define the semi-group of transition operators on $\mathcal{B}$ by

$$(T_t g)(w) = \int_{\mathcal{N}} P(t,w,dy)g(y) \tag{1.7}$$

where $\mathcal{N}=\chi\times N^+$. The T_t have the probabilistic interpretation

$$(T_t g)(w) = E[g(w(t))|W(0)=w] \tag{1.8}$$

where E denotes an expectation. Each $f\varepsilon\mathcal{B}$ induces a linear functional on the space of bounded signed measures. Specifically, we obtain

$$(T_t g,\varphi_o) = \int_{\mathcal{N}}(T_t g)(w)\varphi_o(dw) = E[g(W(t))]. \tag{1.9}$$

In particular, if we choose

$$g_n(w) = \begin{cases} 1 & w\varepsilon\chi\times\{n\} \\ 0 & \text{otherwise} \end{cases} \tag{1.10}$$

we shall have

$$(T_t g_n,\varphi_o) = P[N(t)=n]. \tag{1.11}$$

In the remainder of this paper, we will make repeated use of the standard identity ([6], p. 40)

$$(T_t g)(w) = g(w) + \int_0^t d\tau\int_{\mathcal{N}} P(\tau,w,dy)(Ag)(y) \tag{1.12}$$

which, when W(0) has probability measure φ_o, yields

$$E[g(W(t))] = (g,\varphi_o) + \int_0^t d\tau \int_{\mathcal{N}} P(\tau,dy)(Ag)(y), \tag{1.13}$$

where we have defined $P(\tau,\Lambda)=P[W(\tau)\varepsilon\Lambda]$ and $P(\tau,w,\Lambda)=P[W(\tau)\varepsilon\Lambda|W(0)=\omega]$. As we shall see, the evaluation of (1.13) for a g_n of the form (1.10) and a Markov step process in probability equilibrium is relatively

easy, and sheds light on the properties of the traffic process N.

II. CONDITIONS FOR POISSON COUNTING PROCESSES

We begin by specializing (1.13) for the g_n specified by (1.10); this leads directly to an expression for $P[N(t)=n]$. First, g_n is substituted into (1.1) applied to W, so that we find for $y=(x,k)$

$$(Ag_n)(y) = q(x)Q(x,\Gamma_x)[\delta_{k,n-1}-\delta_{k,n}]. \tag{2.1}$$

Substituting (2.1) into (1.13) yields the basic formulas

$$P[N(t)=0] = 1 - \int_0^t d\tau \int_X q(x)Q(x,\Gamma_x)P(\tau,dx,0) \tag{2.2a}$$

and

$$P[N(t)=n] = \int_0^t d\tau \int_X q(x)Q(x,\Gamma_x)[P(\tau,dx,n-1)-P(\tau,dx,n)] \tag{2.2b}$$

for $n\geq 1$. In the above, we have adopted notation consistent with (1.13); $P(t,\Lambda,n)=P[Z(t)\varepsilon\Lambda,N(t)=n]$. These two equations can be added together and the complement taken to produce the simpler

$$P[N(t)>n] = \int_0^t d\tau \int_X r(x)P(\tau,dx,n) \tag{2.3}$$

in which we have used the more compact notation

$$r(x) = q(x)Q(x,\Gamma_x); \tag{2.4}$$

evidently, $r(x)$ is (intuitively speaking) the rate of the counting process, assuming that the Markov process Z is in state x. Furthermore, $r(Z(t))$ may be thought of as the rate of the counting process N at time t, and its expectation $E[r(Z(t))]$ as the intensity of N at that time. We mention these interpretations at this juncture, because r is in fact a crucial parameter of N.

To relate r to the counting process N, we shall begin by defining some σ-algebras relevant to stationarity and independence properties involving r and N. We define the σ-algebra $\mathcal{N}_t=\sigma\{N(u),u\le t\}$, and observe that $\mathcal{N}_t\subset\mathcal{F}_t=\sigma\{Z(u),u\le t\}$. We note further that the events $\{N(t)-N(s)=n\}\varepsilon\hat{\mathcal{F}}^t=\sigma\{Z(u),u\ge t\}$. With this notation, we claim

Lemma 2.1: Let $\Lambda\varepsilon\hat{\mathcal{F}}^t$. Then almost surely

$$E(\Lambda|\mathcal{N}_t) = E\, E(\Lambda|Z(t))|\mathcal{N}_t] = E[E(\Lambda|\mathcal{F}_t)|\mathcal{N}_t] \tag{2.5}$$

Proof: By the Markov property for Z, $E(\Lambda|Z(t))=E(\Lambda|\mathcal{F}_t)=E(\Lambda|\mathcal{F}_t,\mathcal{N}_t)$. Conditioning again on $\mathcal{N}_t$ and using the fact that the conditioning reduces the right hand expression to $E(\Lambda|\mathcal{N}_t)$ yields (2.5). □

Lemma 2.2: For $0\le s\le t$ we have almost surely

$$E[N(t)-N(s)|\mathcal{N}_s] = \int_s^t E[r(Z(\tau))|\mathcal{N}_s]d\tau \tag{2.6}$$

Proof: If $W_s(u)=(Z(s+u),[N(s+u)-N(s)])$, then W_s is a Markov process with the same statistics as W. We may substitute (2.1) into (1.12) to obtain an expression for $P[N(t)-N(s)=n|Z(s)]$, and proceed as in (2.3) to calculate $P[N(t)-N(s)>n|Z(s)]$. Since $P[N(t)<\infty]=1$, the conditional expectation $E[N(t)-N(s)|Z(s)]$ follows by summing over n; the summation process is valid under the integral sign according to Fubini's theorem. Thus we find

$$E[N(t)-N(s)|Z(s)] = \int_s^t E[r(Z(\tau))|Z(s)]d\tau. \tag{2.7}$$

To reduce (2.7) to (2.6), integrate both sides of (2.7) over $\Lambda\varepsilon\mathcal{N}_s$ with respect to probability measure, using again Fubini's theorem on the right side. Reference to Lemma 2.1 then completes the argument. □

It is easy to calculate the expectation of N from (2.6) or (2.7). Taking the expectation of both sides of either of these equations leads to

$$E[N(t)] = \int_0^t E[r(Z(\tau))]d\tau = \int_0^t (r,\psi_\tau)d\tau. \qquad (2.8)$$

Clearly, (2.8) simplifies if Z is in equilibrium, that is, if $P[Z(t)\varepsilon\Gamma]$ is constant for $t\geq 0$ and any fixed $\Gamma\varepsilon\mathcal{B}$. We note that equilibrium is equivalent to each of the following: (g,ψ_t) is constant, $(T_t g,\psi_o)$ is constant, or $(Ag,\psi_o)=0$ for each $g\varepsilon\mathcal{D}$; this equivalence is easily shown for Feller processes. A more restricted version of the above is given by

<u>Definition 2.3</u>: For a $g\varepsilon\mathcal{D}$, Z is in <u>g-partial equilibrium if</u> (g,ψ_t) is constant for all $t\geq 0$ [or equivalently, if $(T_t g,\psi_o)$ is constant, or $(Ag,\psi_o)=0$]. □

Evidently, g-partial equilibrium is a local steady state condition that permits arbitrary (nonstationary) behavior of parts of the Markov process. In particular, r-partial equilibrium requires that the part of Z relevant to N be in a steady state condition. It may be expected that N experiences homogeneous growth--and is hence a candidate for a Poisson process--if Z is in r-partial equilibrium.

For N to be Poisson, it is also necessary that the future and past of N be independent; more precisely, $[N(t)-N(s)]$ is independent of $\mathcal{N}_s$ for all $0\leq s\leq t$. We shall see that a notion relevant to the appropriate independence is

<u>Definition 2.4</u>: For a $g\varepsilon\mathcal{D}$, Z is g-partially independent of N if, for every $t\geq 0$, $E[g(Z(t))|\mathcal{N}_t]=E[g(Z(t))]$. □

It is clear that $Z(t)$ is independent of $\mathcal{N}_t$ iff Z is g-partially independent of N for every g, and that the independence of $g(Z(t))$ and $\mathcal{N}_t$ is equivalent to g-partial independence if g is an indicator function. With the aid of Definitions 2.3 and 2.4, we can assert conditions under which N is a Poisson process.

Theorem 2.5: N is a Poisson process if and only if

(a) Z is in r-partial equilibrium, and

(b) Z is r-partially independent of N.

Proof: We shall make repeated use of Watanabe's theorem ([5], p. 76), which states that a counting process is Poisson iff

$$E[N(t)-N(s)|\mathcal{N}_s] = \lambda(t-s) \tag{2.9}$$

for $0 \le s \le t$. This is in turn equivalent to the independence of $[N(t)-N(s)]$ and $\mathcal{N}_s$, together with $E[N(t)-N(s)]=\lambda(t-s)$.

Suppose now that (a) and (b) are satisfied. If (b) and (a) are successively applied to the integrand in (2.6), that integrand becomes (r,ψ_o), which we call λ. This proves (2.9), and with it sufficiency. To demonstrate necessity, observe that if N is a Poisson process of intensity λ, the left side of (2.6) becomes $\lambda(t-s)$ according to (2.9). Since r is bounded (as it must be to belong to $\mathcal{B}_A$) and a right continuous step function, we may divide both sides of (2.6) by (t-s) and take limits as $t \searrow s$. The limiting operation yields $E[r(Z(s))|\mathcal{N}_s]=\lambda$ almost surely, whence (b) follows. Taking the expectation of $E[r(Z(s))|\mathcal{N}_t]$ then leads immediately to (a). □

A recent paper concerned with Poisson traffic flows on denumerable state Markovian systems introduces the concept of weak pointwise independence ([13], Section 5), which plays the role of our r-partial independence in our Theorem 2.5. In a sense, weak pointwise independence is a more desirable condition, since it involves only independence of $N(t)$, and not of $\mathcal{N}_t$ (the entire past of N). We shall be able to strengthen our results similarly, referring to $N(t)$ rather than $\mathcal{N}_t$ for necessity and sufficiency.

Definition 2.6: (compare Definition 2.4 and subsequent comments).
For a $g \in \mathcal{G}$, Z is weakly g-partially independent of N if, for every $t \geq 0$, $E[g(Z(t))|N(t)]=E[g(Z(t))]$. □

Theorem 2.7: The conclusion of Theorem 2.5 continue to hold if (b) is replaced by (b*). In particular, (a) and (b*) imply (a) and (b).

Proof: Necessity of (a) and (b*) is obvious, because (b) implies (b*). To demonstrate the sufficiency of (a) and (b*), write (2.2a) and (2.3) in the form

$$P[N(t)=0[= 1 - \int_0^t E[r(Z(\tau))I(\tau,0)]d\tau \tag{2.10}$$

and

$$P[N(t)>n] = \int_0^t E[r(Z(\tau))I(\tau,n)]d\tau \tag{2.11}$$

in which $I(\tau,n)$ is the indicator function of the event $\{N(\tau)=n\}$. That $E[r(Z(\tau))I(\tau,n)]=E[r(Z(\tau))]E[I(\tau,n)]$ is a consequence of (b*), and (a) states that $E[r(Z(\tau))]$ is constant, say equal to λ. Hence, (2.10) and (2.11) become

$$P[N(t)=0] = 1 - \lambda \int_0^t P[N(\tau)=0]d\tau \tag{2.12}$$

and

$$P[N(t)>n] = \lambda \int_0^t P[N(\tau)=n]d\tau. \tag{2.13}$$

We shall deduce from (2.12) and (2.13) that the inter-jump intervals of N are mutally independent and exponentially distributed. To this end, call $T_k = \inf_{t>0}\{t:N(t)=k\}$, and let $S_k = T_{k+1} - T_k$ so that T_n becomes the

sum of inter-jump times, i.e., $T_n = \sum_1^n S_k$. The set equality $\{N(t)=0\}=\{S_1>t\}$, together with the (unique) solution of (2.12) yields as a density function for $T_1=S_1$ the function $f_1(t)=\lambda e^{-\lambda t}$. We now proceed by induction to show that T_n is distributed according to the n-fold convolution of densities f_n.

First, consider that $\{N(t)>n\}=\{T_{n+1}\leq t\}$, so that T_{n+1} has a probability density according to (2.13). From the same equation, using $\{N(t)=n\}=\{T_n\leq t\}-\{T_{n+1}\leq t\}$, we obtain

$$f_{n+1}(t) = \lambda[F_n(t) - F_{n+1}(t)] \qquad (2.14)$$

in which f_n and F_n represent respectively the probability density and probability distribution fucntions of T_n. To complete the inductive argument, we turn to the Laplace transforms $\hat{f}_n$ of the f_n. For T_1, $\hat{f}_1(s)=\frac{\lambda}{s+\lambda}$. Suppose now T_n has generating function $\hat{f}_n(s)=[\frac{\lambda}{s+\lambda}]^n$. Then, using the generating function version of (2.14), we find that $\hat{f}_{n+1}(s)=[\frac{\lambda}{s+\lambda}]^{n+1}$ is the unique solution of (2.14).

The last assertion of the Theorem follows readily: if (a) and (b*) are valid, N is a Poisson process, whence (a) and (b) hold by Theorem 2.5. □

Corollary 2.8: If Z is in probability equilibrium, and if N(t) is independent of Z(t) for each $t\geq 0$, N is a Poisson process.

The Corollary is obvious in view of Theorem 2.7, although the independent increment property of N is easy to prove directly from the independence of N(t) and Z(t). In fact, for any $\Lambda\varepsilon\mathcal{F}^t$, the Markov property yields almost surely

$$P[\Lambda\cap\{N(t)=n\}|Z(t)] = P[\Lambda|Z(t)]P[N(t)=n|Z(t)]. \qquad (2.15)$$

The second term on the right reduces to $P[N(t)=n]$ by virtue of the hypothesis. If we than take the expectation of both sides of (2.15),

we obtain the result $P[\Lambda \cap \{N(t)=n\}]=P(\Lambda)P[N(t)=n]$, as required.

Corollary 2.8 presents global stationarity and independence conditions, whereas the conditions (a), (b) and (b*) appearing in Theorems 2.5 and 2.7 reflect only the local relationships pertinent to N. It is therefore natural that the conditions of Corollary 2.8 are not necessary for N to be a Poisson process. We now give an example in which N is Poisson, but neither of the hypotheses of Corollary 2.8 are satisfied.

A system consists of two exponential servers in tandem, with respective service rates σ_i and queue lengths Z_i. We suppose that the input to the first server is a Poisson stream of intensity λ, with $0<\sigma_2<\lambda<\sigma_1$. We assume further that $Z_2(0)=0$, while $Z_1(0)$ has equilibrium distribution appropriate to an M/M/1 queue with input rate λ and service rate σ_1. As is well known [3], the departure stream N from the first server is Poisson. Nevertheless, $Z=(Z_1,Z_2)$ is not in equilibrium--indeed, Z has only transient states. Moreover, $Z_2(t)=0$ if $N(t)=0$, and in fact, Z_2 depends on N for any initial distribution Z_2.

III. MULTIPLE COUNTING PROCESSES

In this Section we extend the theory of Section II to simultaneous counting processes, as represented by the vector $N=(N_1,N_2,\ldots,N_c)$, each N_i being itself a counting process induced by the Markov process Z. The vector process N is said to be Poisson if each N_i is a Poisson stream, and if the N_i are mutually independent random processes. Our interest lies in finding necessary and sufficient conditions under which N is Poisson.

As before, we define N_i to increase its count by one whenever the step Markov process Z passes directly from x to $\Gamma_x^{(i)}$. In consistency with our former notation (2.4), we shall call

$$r_i(x) = q(x)Q(x,\Gamma_x^{(i)}), \tag{3.1}$$

which is heuristically interpreted as the counting rate of N_i, given that Z is in state x. Conditions that each N_i be a Poisson process are described in terms of the corresponding r_i by Theorem 2.7 via r_i-partial equilibrium and weak r_i-partial independence. However, these individual r_i cannot reveal any information on the mutual independence of the N_i.

Our results require one restriction met by all potential applications of which we are aware. Throughout this Section we shall assume the validity of

Hypothesis 3.1: The N_i are disjoint counting processes if, for each $x\varepsilon\chi$, the $\Gamma_x^{(i)}$, i=1,2,...,c are disjoint sets. □

If now the original Markov process is augmented to W=(Z,N), we have for W (cf. (1.3))

$$\tilde{Q}((x,n),\Gamma\times\{k\}) = \begin{cases} Q(x,\Gamma)\delta_{n+e_i,k} & \Gamma\subset\Gamma_x^{(i)} \\ Q(x,\Gamma)\delta_{n,k} & \Gamma\subset\Gamma_x^{o} \end{cases} \tag{3.2}$$

Here n is the vector count $n=(n_1,n_2,\ldots,n_c)$, e_i is the unit vector along coordinate i, and $\Gamma_x^o = \chi - \bigcup_1^c \Gamma_x^{(i)}$. Note that the disjointness of the $\Gamma_x^{(i)}$ plays a crucial role in the definition of $\tilde{Q}$.

We continue to proceed as in Section I. To compute probabilities of the count, it is convenient to define

$$g_n(w) = \begin{cases} 1 & w\varepsilon\chi\times(n_1,n_2,\ldots,n_c) \\ 0 & \text{otherwise} \end{cases} \tag{3.3}$$

For this g_n and with the benefit of Hypothesis 3.1, we have

$$(Ag_n)(y) = \sum_{i=1}^{c} r_i(x)[\delta_{n+e_i,k} - \delta_{n,k}] \tag{3.4}$$

in which A is the infinitesimal operator of W and $y=(x,k)$. Next, (3.4) is substituted into (1.13) to produce for $\theta=(0,0,\ldots,0)$

$$P[N(t)=\theta] = 1 - \sum_{i=1}^{c} \int_0^t E[r_i(Z(\tau))I(\tau,0]d\tau \tag{3.5}$$

and for $n\neq\theta$

$$P[N(t)=n] = \sum_{i=1}^{c} \int_0^t \{E[r_i(Z(\tau))I(\tau,n-e_i)] - E[r_i(Z(\tau))I(\tau,n)]\}d\tau; \tag{3.6}$$

Here the indicator I is defined as in (2.10) and (2.11), with the proviso that $I(\tau,n-e_i)=0$ if $n_i=0$.

With the aid of these two expressions, we can adapt Theorem 2.7 to *multiple streams*.

Theorem 3.2: N is Poisson, the N_i having respective rates

$$\lambda_i = E[r_i(Z(0)] \tag{3.7}$$

provided that for each i, $i=1,2,\ldots,c$

(a) Z is in r_i-partial equilibrium, and (3.8)

(b*) Z is weakly r_i-partially independent of N. (3.9)

Remark: This Theorem does not claim necessity. Assumption (b*) should be taken to mean that the $r_i(Z(t))$ are each (pairwise) independent of the vector process N at every $t\geq 0$.

Proof: It follows directly from Theorem 2.7 that the respective N_i are Poisson processes whose rates are specified by (3.7). It remains to show that the N_i are mutually independent. To this end, apply the hypotheses of the Theorem to (3.5) and (3.6); these then reduce to

$$P[N(t)=\theta] = 1 - \lambda \int_0^t P[N(\tau)=\theta]d\tau \tag{3.10}$$

and

$$P[N(t)=n] + \lambda \int_0^t P[N(\tau)=n]d\tau = \sum_{i=1}^{c} \lambda_i \int_0^t P[N(\tau)=n-e_i]d\tau, \tag{3.11}$$

where $\lambda=\Sigma\lambda_i$. These equations have as their unique solution

$$P[N(t)=n] = \prod_{i=1}^{c} \{\frac{(\lambda_i t)^{n_i}}{n_i!} \exp[-\lambda_i t]\}, \tag{3.12}$$

so that the $N_i(t)$, i=1,2,...,c are mutually independent for every (fixed) t.

We must further prove that N is a process of independent increments. For this purpose, we define W_u as in the proof of Lemma 2.2, and reiterate the arguments of the proceding paragraph to demonstrate that N(t+u)-N(u) has the same distribution as N(t)=N(t)-N(0). From (3.12), the generating function $\hat{f}$, defined by

$$\hat{f}_{u,v}(s) = E[\exp\{-s^T(N(v)-N(u))\}], \quad 0 \le u \le v \tag{3.13}$$

with N regarded as a column vector and s^T the row vector $s^T=(s_1,s_2,\ldots,s_c)$, is then

$$\hat{f}_{u,v}(s) = \exp\{(v-u)[\Sigma\lambda_i\exp(-s_i) -\lambda]\}. \tag{3.14}$$

The independent increment character of N is now an immediate consequence of the form of $\hat{f}_{u,v}$ (see [12], Sections 37.1 and 37.3).

The hypotheses of Theorem 3.2 are considerably weaker than we might have anticipated. For one thing, the equilibrium requirement (a) is no more stringent than the necessity condition imposed in Theorem 2.5 on each N_i individually. More surprising, however, is the independence assumption (3.9), which requires neither the $N_i(t)$ nor the $r_i(Z(t))$ to bo stochastically independent; rather, the mutual independence of the

$N_i(t)$ (and indeed, the processes N_i) is a natural outcome appearing in the course of the proof.

Nevertheless, we shall assert even weaker sufficiency conditions for N to be Poisson; moreover, the same conditions are also necessary. In particular, the independence hypothesis (b*) in Theorem 3.2 is replaced by the less restrictive

$$\sum_{i=1}^{c} \{E[r_i(Z(t))I(t,n-e_i)] - \lambda_i E[I(t,n-e_i)]\}$$

b'. (3.15)

$$= \{\sum_{i=1}^{c} E[r_i(Z(t))I(t,n)]\} - \lambda E[I(t,n)].$$

for all $n=(n_1,n_2,\ldots,n_c)$ and all $t\geq 0$.

Before proceeding to a proof of sufficiency and necessity, we make three remarks concerning (b') above. First, in the presence of (a) (cf. (3.8)), (b*) (cf. (3.9)) implies that both sides of (3.15) are zero; hence, (b') follows from (b*). On the other hand, for a single stream (b') and (b*) coincide in the presence of (a), as is readily shown by an inductive argument on (3.15). Second, the inductive argument fails for multiple streams (i.e., $c\geq 2$) because successive sums on the left side of (3.15) are incomplete. Indeed, we have been unable to derive any necessity statement claiming stronger independence requirements on N(t) and Z(t). Finally, we observe that necessity has generally been less well understood than sufficiency. We know of no other necessity conditions for multiple processes to be mutually independent Poisson. Even for a single process, the usual sufficiency hypotheses are not also necessary. To illustrate this statement, we observe that detailed balance implies reversibility [9], from which one concludes that a Poisson input in equilibrium leads to a Poisson output

(see [16], or recall the M=M property in [15]). Nevertheless, the converse--necessity--is false; a simple tandem M/M/1 queue has the M=M property, but is neither reversible nor detail balanced.

We now proceed to

Theorem 3.3: N is Poisson if and only if (a) and (b') hold.

Proof: To show sufficiency, we simplify (3.5) and (3.6) by introducing (a) and (b'). The resulting relations are (3.10) and (3.11), so that the remainder of the argument follows the proof of Theorem 3.2 verbatim.

If N is Poisson, the necessity part of Theorem 2.7 applies to each N_i, so (a) holds, the respective rates λ_i being specified by (3.7). Now differentiate (3.5) and (3.6), observing that the integrands are continuous from the stochastic continuity of Z. Differentiation thus yields the single set of equations

$$\frac{d}{dt}\{E[I(t,n)]\}= \sum_{i=1}^{c}\{E[r_i(Z(t))I(t,n-e_i)]-E[r_i(Z(t))I(t,n)]\}. \tag{3.16}$$

But the probability $P[N(t)=n]=E[I(t,n)]$ is furnished by (3.12), from which

$$\frac{d}{dt}\{E[I(t,n)]\}= \sum_{i=1}^{c} \lambda_i\{E[I(t,n-e_i)]-E[I(t,n)]\}. \tag{3.17}$$

Elimination of the derivative between (3.16) and (3.17) then verifies (b'). □

Let us consider briefly the new condition (for all n)

$$\sum_{i=1}^{c} E[r_i(Z(t))|N(t)=n] = \lambda \quad \text{and}$$

b^+. (3.18)

$$\sum_{i=1}^{c}\{E[r_i(Z(t))I(t,n-e_i)] - \lambda_i E[I(t,n-e_i)]\} = 0.$$

Some implications involving (b^+) are: (a) and $(b^*) \Rightarrow (b^+) \Rightarrow (a)$ and (b').[1] Therefore, (b^+) suffices for N to be Poisson, but we have been unable to prove that (b^+) is necessary. Our best result in that direction is an asymptotic one, namely

Corollary 3.4: If N is Poisson

$$\sum_{i=1}^{c} E[r_i(Z(t))|N(t)=n]-\lambda = 0(t^{-1}) \quad \text{as } t\to\infty \tag{3.19}$$

for all n.

Remark: It is trivial that both sides of (3.15) tend toward zero as t tends to infinity; this occurs becuase $P[N(t)=n]\to 0$ (any n) if N is Poisson.

Proof of Corollary: We return to property (b') (i.e., (3.15), which is valid under the stated hypothesis. This equation is rewritten in terms of conditional expectations as

$$\begin{aligned} &\sum_{i=1}^{c} \left(\frac{n_i}{\lambda_i}\right)\{E[r_i(Z(t))|N(t)=n-e_i] - E[r_i(Z(t))]\} \\ &\quad = t\{\sum_{i=1}^{c} E[r_i(Z(t))|N(t)=n] -\lambda\}, \end{aligned} \tag{3.20}$$

in which we have also used (3.12) to express the ratio between $P[N(t)=n-e_i]$ and $P[N(t)=n]$, and have been able to assign arbitrary values to probabilities conditioned on $\{N(t)=n-e_i\}$ when $n_i=0$. Since the r_i are necessarily bounded because they are in the range of the infinitesimal operator, the left side of (3.20) is also bounded in t for each fixed n. Hence (3.19) follows. □

We conclude our discussion by observing that the necessarily

[1]Note that the second equality of (b^+) contains (a). Take $n=ke_j$, so that only the j terms remain. Then sum over k; the result is $E[r_j(Z(t))]=\lambda_j$, as required.

infinite sets of equalities established thus far can be simplified--at least in principle--through the introduction of generating functions. To this end, we determine the generating function version of the infinite set of integral equations (3.5) and (3.6) under the assumption that N is Poisson as previously supposed. Actually, it is easier to work with the differential equations (3.16), multiplying both sides by $y = \Pi y^{n_i}$ and summing over n. For convenience, we let $z_i = 1 - y_i$, and confine ourselves to $z = (z_1, z_2, \ldots, z_c)$ satisfying $|1 - z_1| < 1$ for each i. We take

$$\hat{G}_t(z) = E\left[\prod_{i=1}^{c} (1-z_i)^{N_i(t)}\right] \qquad (3.21)$$

which equals $\exp[-t\Sigma(\lambda_i z_i)]$ by the Poisson assumption, so that the time derivative of $\hat{G}_t(z)$ becomes $\Sigma \lambda_i z_i \hat{G}_t(z)$. Further, we define

$$\hat{G}_{tj}(z) = E\left[\prod_{i=1}^{c} (1-z_i)^{N_i(t)} \middle| r_j(Z(t))\right]. \qquad (3.22)$$

With this notation, the infinite set of equations (3.16) [equivalent to (3.5) and (3.6)] becomes

$$\sum_{i=1}^{c} \lambda_i z_i [\hat{G}_{ti}(z) - \hat{G}_t(z)] = 0. \qquad (3.23)$$

The significance of this equation is recognized in

<u>Corollary 3.5</u>: Condition (b') of Theorem 3.3 is equivalent to (3.23).□

Although (3.23) "looks neater" than (b') [cf. (3.15)], it is no more susceptible to simplification. It is only in the case of the single stream (i.e., c=1), that (3.23) [and for that matter, conditions (b*), (b') or (b^+)] reduce to the independence statement of Theorem 2.7, which in combination with (a) is equivalent to a Poisson N.

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